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Covariance and Quantum Logic

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Abstract Considering the fundamental role symmetry plays throughout physics, it is remarkable how little attention has been paid to it in the quantum-logical literature. In this paper, we discuss G-test spaces—that is, test spaces hosting an action by a group G—and their logics. The focus is on G-test spaces having strong homogeneity properties. After establishing some general results and exhibiting various specimens (some of them exotic), we show that a sufficiently symmetric G-test space having an invariant, separating set of states with affine dimension n, is always representable in terms of a real Hilbert space of dimension $n+1$, in such a way that orthogonal outcomes are represented by orthogonal unit vectors.

1 Introduction

In the most basic formulation of quantum mechanics, the set of directly testable propositions concerning a physical entity form, not a Boolean algebra, but the projective geometry $\mathbb{P}(\mathbf{H})$ of closed subspaces of a Hilbert space. This is an orthomodular—in finite dimensions, modular—orthocomplemented lattice, and thus, in a sense, "locally boolean". Gleason's theorem tells us that the states of the quantum system are in one-to-one correspondence with probability measures defined on $\mathbb{P}(\mathbf{H})$. From this, one can reduce essentially the entire apparatus of quantum theory to a non-classical probability theory in which boolean algebras are replaced by projective geometries [5, 7].

Many efforts have been made to motivate this framework for a generalized probability theory in an autonomous way. These typically begin with a spare "operational" infrastructure, from which—after the imposition of various (and variously motivated) axioms—an ordered set of "propositions" is constructed, and shown to have the structure of, say, an orthomodular poset, or an orthomodular lattice, *etc*. We use the phrase *quantum logic* to refer

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to such accounts generically. A particularly elegant example is the theory of *test spaces*, or *manuals*, due to D.J. Foulis and the late C.H. Randall [1–3]. In this approach, the basic structure to be generalized is not the ordered set of subspaces, but rather, the set $\mathfrak{F}_{\mathbf{H}}$ of *frames*—unordered orthonormal bases—of $\mathbf{H}$. In quantum probability theory, each frame of $\mathbf{H}$ represents the outcome-set of a physical experiment. Accordingly, Foulis and Randall begin with an abstract collection $\mathfrak{A}$ of non-empty sets, called *tests*, each considered to be the outcome-set for some statistical measurement. Any test space $\mathfrak{A}$ is associated in a natural way with an ordered set called the *logic* of $\mathfrak{A}$, denoted by $\Pi(\mathfrak{A})$. Where $\mathfrak{A} = \mathfrak{F}_{\mathbf{H}}$, this logic is isomorphic to the subspace lattice $L(\mathbf{H})$. For another example, where $\mathfrak{A}$ consists of all partitions of a set S by sets belonging to an algebra $\mathcal{B}$ on S, $\Pi(\mathfrak{A}) \simeq \mathcal{B}$.

Considering the fundamental role symmetry plays throughout physics, and particularly in quantum theory, it is remarkable how little it has been exploited in the quantum-logical literature. In particular, the standard quantum logic, that is, the projection lattice $L(\mathbf{H})$, and its attendant frame manual $\mathfrak{F}(\mathbf{H})$, are strikingly symmetrical objects. In several previous papers [9–11], the second author has attempted to lay a foundation for a theory of *symmetric* test spaces, that is, test spaces that are in some sense homogeneous under a group action. In particular, Wilce [11] introduced the notion of a *fully symmetric* G-test space. This is a test space in which any two tests have the same cardinality, and in which any bijection between two tests is implemented by an element of G. The motivating example, of course, is the frame manual $\mathfrak{F}(\mathbf{H})$, with $\mathbf{H}$'s unitary group playing the role of G: any bijection between two frames extends (uniquely, at that) to a unitary operator.

The present paper continues the study of symmetric and fully symmetric test spaces. After reviewing, in Sect. 2, some essential background information on test spaces and their logics, in Sect. 3 we produce many examples of fully symmetric test spaces (some of them fairly exotic), and present a general construction whereby two fully symmetric test spaces can be combined to yield another. In Sect. 4, we note that every fully symmetric G-test space arises in a natural way from a triple (G, K, σ) consisting of a group G, a subgroup $K \leq G$, and an element $\sigma \in G \setminus K$. We then show that $\mathfrak{A}$ is connected (as a hypergraph) iff it is fully symmetric under the subgroup generated by K and σ. Finally, in Sect. 5 we show that any test space having a fairly weak homogeneity property relative to a *compact* group, and having a finite-dimensional, separating set of states, can be represented covariantly as a space of orthonormal subsets (not necessarily bases!) of a finite-dimensional Hilbert space.

2 Test Spaces and Orthoalgebras

In this section, we briefly review some basic lore concerning abstract quantum logics (orthomodular lattices and posets, and, more generally, orthoalgebras), and the Foulis-Randall construction of such objects as invariants—"logics"—of test spaces. For details, see [1, 10].

2.1 Quantum Logics

An *orthoalgebra* is a structure $(L, \oplus, 0, 1)$ consisting of a set L, two distinguished elements 0 and 1, and a commutative, associative, cancellative partial operation $\oplus$ such that, for all $a \in L$,

(1) $a \oplus 0 = a$;
(2) $\exists\, a' \in L$ with $a \oplus a' = 1$;
(3) $a \oplus a$ exists only if $a = 0$.

An orthoalgebra L can be partially ordered by setting

$$a \le b \quad \Leftrightarrow \quad \exists c \in L \text{ with } b = a \oplus c.$$

The mapping $a \mapsto a'$ is an orthocomplementation with respect to $\le$, and $a \oplus b$ is defined iff $a \perp b$, i.e., $a \le b'$. If $a \perp b$, then $a \oplus b$ is a *minimal* (but not necessarily least) upper bound for $a, b \in L$.

Any orthomodular poset—hence, any orthomodular lattice, hence,any Boolean algebra—can be regarded as an orthoalgebra with $a \oplus b = a \vee b$ provided $a \le b'$. The given order then coincides with the one defined above. Orthoalgebras arising in this way from OMPs have a nice characterization, as follows: if $(L, \oplus, 0, 1)$ is an orthoalgebra, then the associated orthoposet $(L, \le, ')$ is an OMP iff, wherever defined, $a \oplus b$ is the *least* upper bound of $a, b \in L$. This in turn is equivalent to the condition that if a, b, c are pairwise orthogonal in L, then $(a \oplus b) \oplus c$ exists. This last condition is called *ortho-coherence*. Hence, OMPs are essentially the same things as orthocoherent OAs; accordingly, orthomodular lattices are the same things as lattice-ordered orthoalgebras.

2.2 Test Spaces

Orthoalgebras arise from simpler combinatorial objects called *algebraic test spaces*, or—to revive an older term—*manuals*. A *test space* is a pair $(X, \mathfrak{A})$ consisting of a set X and a covering $\mathfrak{A}$ of X by non-empty subsets, called *tests*. These nay be understood as the outcome sets for various "experiments"; accordingly, subsets of tests are called *events*. A *state* on a test space $(X, \mathfrak{A})$ is a mapping $\omega : X \to [0, 1]$ summing to one over each test.

Discrete classical probability theory concerns states on a *classical* test space, that is, a test space $(E, \{E\})$ consisting of a single test. Elementary quantum probability theory concerns states on a *frame manual*, i.e., a test space $(X_{\mathbf{H}}, \mathfrak{F}_{\mathbf{H}})$, where $X_{\mathbf{H}}$ is the unit sphere, and $\mathfrak{F}_{\mathbf{H}}$, the set of unordered orthonormal bases, of a Hilbert space $\mathbf{H}$. Gleason's theorem tells us that so long as $\dim(\mathbf{H}) > 2$, every state on $(X, \mathfrak{F})$ arises from a density operator on $\mathbf{H}$ via the "Born rule" $\omega(x) = \mathrm{Tr}(\rho P_x)$. Note that both examples are uniform (the former, trivially).

More sophisticated models of both classical and quantum probability theory also fall within the descriptive scope of the theory of test spaces. Indeed, if L is any orthoalgebra (in particular: any boolean algebra, any projection lattice, any orthomodular lattice or poset…), then we can consider the test space $\mathfrak{A}_L$ consisting of finite (or countable, or all) orthopartitions of the unit; the states on this correspond in an obvious way to finitely-(or countably, or totally) additive probability measures on L.

2.3 The Logic of a Test Space

An *event* of a test space $(X, \mathfrak{A})$ is a subset of a test. We write $\mathcal{E} = \mathcal{E}(X, \mathfrak{A})$ for the set of all events. Events $A, B \in \mathcal{E}$ are *complementary* (written $A\,\mathbf{co}\,B$) iff they partition a test, and *perspective* (written $A \sim B$) iff they have a common complementary event. A test space $(X, \mathfrak{A})$ is *algebraic*, or a *manual*, iff perspective events have *the same* complementary events—that is, if

$$A \sim B\,\mathbf{co}\,C \quad \Rightarrow \quad A\,\mathbf{co}\,C$$

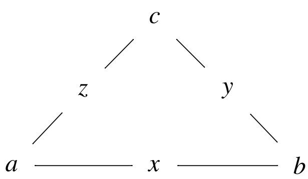
for all events $A, B, C \in \mathcal{E}$: If $(X, \mathfrak{A})$ is algebraic, then perspectivity is an equivalence relation on $\mathcal{E}$. Write $[A]$ for the equivalence-class of A. The quotient set $\Pi := \mathcal{E}/\sim$ carries a well-defined partial operation given by $[A] \oplus [B] := [A \cup B]$ whenever A and B are disjoint events with $A \cup B \in \mathcal{E}$. Defining $0 := [\emptyset]$ and $1 = [E]$, where E is any test, we obtain an orthoalgebra $(\Pi, \oplus, 0, 1)$, called the *logic* of $(X, \mathfrak{A})$.

By way of illustration, if $X = E$ and $\mathfrak{A} = \{E\}$, then $\Pi \simeq \mathcal{P}(E)$; if $(X, \mathfrak{F})$ is the frame manual of a Hilbert space $\mathbf{H}$, then tests are orthonormal bases, events are orthonormal sets, and two events are perspective iff they have the same closed span. It follows that $\Pi \simeq L(\mathbf{H})$. In fact, any orthoalgebra L arises as the logic of a test space (for X, take the set of non-zero elements of L, and for $\mathfrak{A}$, the set of all finite subsets of X orthosumable to 1).

2.4 Greechie Test Spaces

A test space $(X, \mathfrak{A})$ is *Greechie* iff (i) $|E| \geq 3$ for all $E \in \mathfrak{A}$, and (ii) $|E \cap F| \leq 1$ for all $E, F \in \mathfrak{A}$. *Remarks:* (i) The three-dimensional *projective* frame manual is Greechie. (ii) Any Greechie test space is algebraic by default.

It is often useful to represent small Greechie test spaces by so-called *Greechie diagrams*, in which each outcome is represented by a node, and in which the nodes belonging to a test are joined by a line or some other smooth arc. For example, consider the Greechie test space $(X, \mathfrak{A})$ where $X = \{a, b, c, x, y, z\}$ and $\mathfrak{A} = \{\{a, x, b\}, \{b, y, c\}, \{c, z, a\}\}$: this has the following Greechie diagram:



Note that the events $A = \{a, z\}$ and $B = \{b, y\}$ are perspective; hence $[A] = [B] =: p \in \Pi(X, \mathfrak{A})$. The outcomes a and b are orthogonal, and $[a], [b] \leq p$; however, $[a] \oplus [b] = [\{a, b\}] \not\leq p$. Hence, the logic of $(X, \mathfrak{A})$ is an orthoalgebra that is not an OMP.

In fact, this example is prototypic of a test space with a non-orthocoherent logic. A *k-loop* in a test space $(X, \mathfrak{A})$ is a finite sequence $E_1, E_2, \ldots, E_k$ of tests with $E_i \cap E_{i+1} \neq \emptyset$ for $i = 1, \ldots, k-1$ and with all other intersections empty, save that $E_1 \cap E_k \neq \varnothing$. In this language, the test space pictured above consists of a single 3-loop. In fact, the 3-loop is the only obstruction to a Greechie test space's having an orthocoherent logic, and a 4-loop is the only further obstruction to that logic's being a lattice (this is the celebrated *Loop Lemma* of Greechie [4].)

3 Fully Symmetric Test Spaces

Let G be a group. A G**-test space** is a test space $(X, \mathfrak{A})$ equipped with an action of G on X such that, for every test $E \in \mathfrak{A}$ and every $a \in G$, $a(E) \in \mathfrak{A}$ as well. We say that $(X, \mathfrak{A})$ is G*-symmetric* [9, 10] iff G acts transitively on $\mathfrak{A}$, and the stabilizer of a test $E \in \mathfrak{A}$ acts transitively on E. Note that a G-symmetric test space is necessarily *uniform*, meaning that all tests have the same cardinality, which we then call the *rank* of the test space.

Definition 3.1 A uniform G-test space $(X, \mathfrak{A})$ is *fully symmetric* iff, for any bijection $f : E \to F$ between two tests $E, F \in \mathfrak{A}$, there exists some $a \in G$ with $f(x) = ax$ for all $x \in E$. If this group element a is always unique, we say that $(X, \mathfrak{A})$ is *strongly symmetric*. By a *fully* *symmetric* (resp. *strongly*) *symmetric* test space, we mean one fully or strongly symmetric under some group action, or, equivalently, under its automorphism group.

Remark Evidently, every fully symmetric test space is symmetric. Note, too, that for a G-test space $\mathfrak{A}$ to be fully symmetric, it suffices that (i) G act transitively on the tests of $\mathfrak{A}$, and (ii) for some (hence, any) test $E \in \mathfrak{A}$, the stabilizer G_E act on E as the full symmetric group S_E of all permutations of E. For later reference, we shall summarize this last situation by saying that G_E *acts fully* on E. More generally, we shall say that a subgroup J of G acts fully on E iff every permutation of E can be implemented by an element of J (even if E is not invariant under J). Note that, to establish that this is the case, it suffices to show that J contains an element σ that permutes the elements of E cyclically, and an element τ that transposes two elements of E, leaving the others fixed.

Example 3.2 A classical test space $\{E\}$ is fully (indeed, strongly) symmetric under the symmetric group S_E.

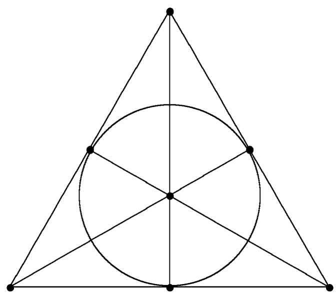
Example 3.3 The frame manual of a Hilbert space is strongly symmetric with respect to that space's unitary group, since any bijection between two frames uniquely determines a unitary operator on $\mathbf{H}$. The *projective frame manual*, in which tests are represented by maximal orthogonal sets of one-dimensional subspaces, is fully, but not strongly, symmetric under $\mathbf{H}$'s unitary group.

Example 3.4 (Uniform Partitions) Let S be a finite set of size $|X| = nk$; let X denote the set of k-element subsets of S, and let $\mathfrak{A}$ consist of all partitions of S into n blocks of k elements. Then $(X, \mathfrak{A})$ is a fully symmetric algebraic test space of rank n. (Such a test space typically has four-loops, so its logic isn't an OML.)

Example 3.5 (Grids) An n-by-n grid can be regarded as the Greechie diagram of a test space (with outcomes corresponding to the vertices, and tests, to the rows and columns). As this test space has 4-loops but no 3-loops, it has an orthocoherent, but not lattice-ordered, logic. (Note, too, that the state-space of this test space is essentially the set of doubly-stochastic n-by-n matrices.) Such a test space is fully symmetric under the subgroup of $S(n \times n)$ generated by $S_n \times S_n$, together with the bijection (transposition) that exchanges the two factors.

Note that an $n \times n$ grid arises as a sub-test space of a uniform test space of partitions (the underlying set being essentially the set of $n \times n$ permutation matrices).

Example 3.6 (Projective Planes) The projective plane of order 2, the famous *Fano plane*, is pictured below.



Notice that this is a Greechie diagram! The corresponding test space is very odd: every outcome is orthogonal to every other, yet the logic, far from being Boolean, is not even orthocoherent. (Orthoalgebras of this sort, termed *centeria*, are discussed in [6].) Notwithstanding these strange features, this test space is fully symmetric relative to its collineation group. These remarks apply to any projective plane.

Example 3.7 (Platonic solids) Let X denote the set of *edges* of a regular polyhedron P; let $\mathfrak{A}$ consist of sets of edges meeting at a vertex. Two sets in $\mathfrak{A}$ meet in at most one edge, so $(X, \mathfrak{A})$ is Greechie. If P is a platonic solid, $(X, \mathfrak{A})$ is fully symmetric relative to the group of rigid motions of P. Note that each face of P yields a loop in $\mathfrak{A}$, and that these are the *shortest* loops $\mathfrak{A}$. Thus, for instance, the logic of the tetrahedron, octohedron and icosohedron are non-orthocoherent orthoalgebras, that of the cube is an OMP, but not a lattice, and the logic of the dodecahedron is an OML.

Example 3.8 (Spaces of simplices) For another example, let $\mathfrak{A}$ consist of the vertex sets of all regular unit tetrahedra in $\mathbb{R}^3$: since the set of symmetries of a simplex is the fully symmetry group on the vertices, and since the rotation group plainly maps any regular tetrahedron into any other, this is a fully—indeed, strongly—$O(3)$-symmetric test space. Since distinct tetrahedra centered at the origin can share at most one vertex, this test space is Greechie, hence, algebraic. Similar examples can be obtained using regular unit $n+1$-simplices in $\mathbb{R}^n$.

3.1 Combinations of Fully G-Symmetric Test Spaces

Many more examples of fully-symmetric test spaces can be obtained by means of the following construction.

Definition 3.9 If $(X, \mathfrak{A})$ and $(Y, \mathfrak{B})$ are test spaces, let $B(\mathfrak{A}, \mathfrak{B})$ denote the set of all bijections $f : E \to F$ where $E \in \mathfrak{A}$ and $F \in \mathfrak{B}$. Identifying such a bijection with its graph $f \subseteq E \times F$, we regard $(X \times Y, B(\mathfrak{A}, \mathfrak{B})$ as a test space. It can be shown that $(X \times Y, B(\mathfrak{A}, \mathfrak{B})$ is the direct product of X and Y in the category of uniform rank-n test spaces [9].

Suppose now that $(X, \mathfrak{A})$ and $(Y, \mathfrak{B})$ are respectively G- and G'-symmetric test spaces. Let $G \times G'$ act on $X \times X'$ in the usual way. Then we have

Theorem 3.10 *If $(X, \mathfrak{A})$ and $(Y, \mathfrak{B})$ are fully-symmetric G- and G'-test spaces, respectively, of common rank n. Then $(X \times Y, \mathfrak{B}(\mathfrak{A}, \mathfrak{B}))$ is fully $G \times G'$-symmetric.*

Proof Let $f, g \in B(\mathfrak{A}, \mathfrak{B})$, say with $f : E \to F$ and $g : E' \to F'$. Let $\phi : f \to g$ be a bijection. We must produce elements $a, b \in G$ such that $\phi(x, f(x)) = (ax, bf(x))$ for every $x \in E$. Now, if $\pi_1 : E' \times F' \to E'$ is the first coordinate projection, so that $\pi_1(x', y') = x'$ for all $x', y' \in E \times F$, then (since f, ϕ and g are bijections), the mapping

$$x \mapsto \pi_1(\phi(x, f(x)))$$

is a bijection from E to E'. Thus, there exists some $a \in G$ with $\pi_1(\phi(x, f(x))) = ax$. Similarly, $y \mapsto \pi_2(\phi(f^{-1}(y), y))$ is a bijection from F to F', hence, implemented by some $b \in G'$. Thus, $\phi(x, f(x)) = (ax, bf(x))$. ◻

4 Constructing G-Test Spaces

As discussed in [11], the structure of a fully symmetric G-test space can be recovered from purely group-theoretic data. Indeed, given a group G and a pair of subgroups $H, K \leq G$, let $X = G/K$, and let $\mathfrak{G}(H, K) := \{a\{hK|h \in H\}|a \in G\}$: this gives us a covering of X by non-empty subsets, so we may regard $(G/K, \mathfrak{G}(H, K))$ as a test space, which is easily seen to be symmetric under the natural action of G. Conversely, given a fully symmetric test space $(X, \mathfrak{A})$, choose a test $E_o \in \mathfrak{A}$ and an outcome $x_o \in E_o$: taking $K \leq G$ to be the stabilizer of x_o and $H \leq G$ be *any* subgroup of the stabilizer of G that acts fully on E_o, one finds that $(X, \mathfrak{A})$ is isomorphic to $(G/K, \mathfrak{G}(H, K))$.

In fact, one can reconstruct $(X, \mathfrak{A})$ using slightly less material. Choosing $x_o \in E_o \in \mathfrak{A}$ as above, and again let K denote the stabilizer of x_o. Now let σ be any element of G acting on E_o as a cyclic permutation. (Such an element must exist, since $(X, \mathfrak{A})$ is fully symmetric.) Since any transposition on $E \setminus \{x_o\}$ is represented by an element of K, the subgroup K and the group element σ together generate a group $\langle K, \sigma\rangle$ that acts fully on E_o. It follows that the data (G, K, σ) entirely fix the structure of $(X, \mathfrak{A})$.

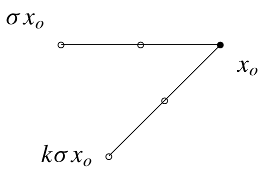
It may be helpful to bear in mind the example of the frame manual $(X, \mathfrak{F})$ associated with $\mathbf{H} = \mathbb{R}^3$. This is fully symmetric under the orthogonal group $O(3)$. If we let $x_o = (0, 0, 1)$ be the "north pole" of the unit sphere, the stabilizer K can be identified with $O(2)$. In the preceding construction, the sphere is identified with $O(3)/O(2)$; $E_o = \{x_o, x_1, x_2\}$ is a fixed (but arbitrary) orthonormal basis extending x_o, and σ is an orthogonal matrix permuting these elements cyclically—that is, a $2\pi/3$ rotation about $x_o + x_1 + x_2$.

In what follows, we work with an arbitrary but fixed group G, and fully symmetric G-test space $(X, \mathfrak{A})$, and we assume that $x_o \in E_o \in \mathfrak{A}$, $K \leq G$, and $\sigma \in G \setminus K$ have been chosen as above.

Lemma 4.1 *For all $x \in X$, $x \perp x_o$ iff $x = ax_o$ for some $a \in K\sigma K$.*

Proof If $x \perp x_o$, then there exists a test E containing both x and x_o. Let $f : E \to E_o$ be any bijection with $f(x_o) = x_o$ and $f(x) = \sigma x_o$, and let $a \in G$ represent f, so that $ax_o = x_o$ and $ax = \sigma x_o \in E_o$. Since $ax_o = x_o$, we have $a \in K$, whence, $a^{-1}\sigma \in K\sigma K$. For the converse, let $a = k\sigma k' \in K\sigma K$. Then, since $\sigma x_o \perp x_o$, we have $k\sigma k' x_o = k\sigma x_o \perp kx_o = x_o$. □

This tells us, in effect, that we can *pivot* any outcome $x \in x_o^\perp$ into a standard position—namely, σx_o—while holding x_o fixed. Equivalently, every point orthogonal to x_o has the form $k\sigma x_o$ where $k \in K$, as illustrated below.



Corollary 4.2 $ax_o \perp bx_o$ *iff* $b^{-1}a \in K\sigma K$ *iff* $\sigma \in Kb^{-1}aK$.

Corollary 4.3 $K\sigma K = K\sigma^{-1}K$.

Proof As $\sigma^{-1}x_o \perp x_o$, $\sigma^{-1} \in K\sigma K$; hence, $K\sigma^{-1}K = K\sigma K$. □

Remark

(1) The double cosets KaK partition G. Thinking of X as G/K, this is a coarsening of the partition of X by left cosets. Indeed, KaK is just the union of the orbit of aK in X under the action of K. In the example discussed above where $X = O(3)/O(2)$ is the unit sphere in $\mathbb{R}^3$, with K representing the point x_o (the north pole), the double coset KaK would represent the line of latitude containing the point ax_o. Lemma 4.1 says, in this instance, that $ax_o \perp x_o$ iff ax_o lies on the equator.
(2) With the preceding example in mind, we may wish to think of the set of double cosets KgK as forming a kind of scale of "angles" between outcomes, with $KeK = K$ corresponding to 1 and $K\sigma K$ corresponding to 0. We will not pursue this further, except to note that the mapping $aK, bK \mapsto Kb^{-1}aK$ is well-defined, and in some respects formally resembles an inner product.

4.1 Orthoconnectedness

Call two outcomes x and y of $(X, \mathfrak{A})$ *n-orthogonal* iff there exists a sequence of outcomes $(x_o, \dots, x_n)$ in X^n with

$$x_o = x \perp x_1 \perp \cdots \perp x_n = y.$$

Note that we allow repetitions, so n-orthogonality implies $n + k$ orthogonality for every k. Every $x \in X$ is 0-orthogonal to itself, and two elements of X are 1-orthogonal iff they are orthogonal. Distinct non-orthogonal outcomes x and y are 2-orthogonal iff there exists a third distinct element z with $x \perp z \perp y$.

By Lemma 4.1, $y \perp x_o$ iff $y = ax_o$ where $a \in K\sigma K$. If $x_o \perp z \perp y$, then $z = ax_o$ as above, and y can be obtained from x_o by a bijection fixing z, so, by a group element $b \in G_z = aKa^{-1}$. Thus, $b \in (K\sigma K)K(K\sigma^{-1}K) = K\sigma K\sigma^{-1}K$. By Corollary 3.3, $K\sigma^{-1}K = K\sigma K$, so

$$b \in K\sigma K\sigma K = (K\sigma K)^2.$$

More generally, we have the

Theorem 4.4 *bx_o is n-orthogonal to x_o iff $b \in (K\sigma K)^n$.*

Proof We proceed by induction on n. From the foregoing discussion, we already have the base case (as well as the case $n = 2$). Suppose the statement of the theorem holds for n-orthogonality, and let $y = bx_o$ be $n + 1$-orthogonal to x_o. Then there exists some chain

$$x_o \perp x_1 \perp \cdots \perp x_n \perp y = bx_o.$$

Let $x_1 = ax_o$ where $a = k\sigma$ with $k \in K$. Then we have

$$x_o = a^{-1}x_1 \perp a^{-1}x_2 \perp \cdots \perp a^{-1}y = a^{-1}bx_o.$$

Thus, $a^{-1}bx_o$ is n-orthogonal to x_o, whence, by hypothesis, $a^{-1}b \in (K\sigma K)^n$. Thus, $b \in a(K\sigma k)^n \subseteq K\sigma(K\sigma K)^n = (K\sigma K)^{n+1}$. □

We say that $(X, \mathfrak{A})$ is *orthoconnected* iff, for every pair of outcomes $x, y \in X$ is n-orthogonal for some integer n. If every pair of outcomes is k-orthogonal for some $k \leq n$, we shall say that $(X, \mathfrak{A})$ is *n-orthoconnected.* (Thus, a 0-orthoconnected test space has just a single outcome, and a 1-orthoconnected test space is classical.)

Corollary 4.5 *If* $(X, \mathfrak{A})$ *is* n*-orthoconnected, then* $G = (K\sigma K)^n$. *In particular, the latter is a group.*

By a *chain* in a test space $(X, \mathfrak{A})$, we mean a finite sequence $(E_o, E_1, \ldots, E_n)$ of tests with $E_i \cap E_{i+1} \neq \emptyset$. Evidently, distinct outcomes x and y are n-orthogonal iff there exists a chain $E_1, \ldots, E_n$ with $x \in E_1$ and $y \in E_n$. It follows that $(X, \mathfrak{A})$ is orthoconnected iff there exists a chain between any two tests.

Proposition 4.6 *Let* $(X, \mathfrak{A})$ *be a fully* G*-symmetric test space of rank* ≥ 3. *Let* $x_o \in E_o \in \mathfrak{A}$, *and let* J *be any subgroup of* G *containing* $K = G_{x_o}$ *and acting fully on* E (*so that* $J \leq H$, *the stabilizer of* E_o). *Then* $(X, \mathfrak{A})$ *is orthoconnected iff it is fully* J*-symmetric.*

Proof Let $\mathfrak{A}_o$ denote the *chain component* of E_o, i.e., the set of tests reachable from E_o by a chain. This is both K- and H-invariant, whence, J-invariant. Thus, if J acts transitively (much less fully) on $\mathfrak{A}$, then $\mathfrak{A}_o = \mathfrak{A}$. For the converse, note that as J acts as the full permutation group of E_o, we need only show that each $E \in \mathfrak{A}$ has the form cE_o for some $c \in J$. Let $n(E)$ denote the length of the shortest chain from E_o to E. If $n(E) = 0$, then $E = E_o$, and there is nothing to show. Assume that every test E' with $n(E') \leq n$ is reachable from E_o by some $a \in J$. Suppose that $n(E) = n + 1$. Then there exists a test $E' \in \mathfrak{A}$ with $n(E') = n$, such that $E' \cap E \neq \emptyset$. Let $x' \in E' \cap E$. Let $b \in G$ take E' bijectively to E, with $bx' = x'$, and let $a \in J$ with $aE_o = E'$. Without loss of generality, we can assume that $ax_o = x'$ (since J acts fully on E_o). Then $G_{x'} = aKa^{-1} \subseteq J$. Since $b \in G_{x'}$, $b \in J$, and thus $c := ba \in J$ takes E_o to E. □

In Proposition 4.6 we can take J to be the entire subgroup generated by K and the stabilizer $H = G_{E_o}$ of E_o, or that generated by K and any group element σ acting as a cyclic permutation of E_o.

Corollary 4.7 *Let* $(X, \mathfrak{A})$ *be fully* G*-symmetric, of rank* ≥ 3. *Then* $(X, \mathfrak{A})$ *is orthoconnected iff it is fully symmetric under* $\langle G_x, G_y \rangle$ *for any* $x \perp y$ *in* X.

Proof Suppose that x and y are orthogonal outcomes belonging to a test $E \in \mathfrak{A}$. By Proposition 4.6, it suffices to show that $\langle G_x, G_y \rangle$ acts fully on E. For this, it is enough to note that every transposition of elements of E belongs to $\langle G_x, G_y \rangle$. Every transposition on E fixes x or y, save for the transposition (x, y) that interchanges x and y; but since $|E| \geq 3$, this last is a product of transpositions fixing x and y (specifically, $(x, y) = (x, z)(y, z)(x, z)$, where $z \in E \setminus \{x, y\}$). □

Remark It is geometrically evident that the frame manual $\mathfrak{F}_3$ of a three-dimensional Real Hilbert space is orthoconnected (indeed, 2-orthoconnected). As an illustration of Corollary 4.7, note that the orthogonal group $O(3)$ is generated by rotations about any two orthogonal axes.

5 A Linear Representation

As the examples adduced in Sect. 2 illustrate, fully symmetric test spaces can be very unlike the frame and projection manuals arising in quantum mechanics. There is, however, a sense in which any fully symmetric test space having a finite-dimensional, separating set of states supports a covariant Hilbert space interpretation. To put what follows into perspective, note that it is always possible, given a test space $(X, \mathfrak{A})$, to construct a unitary representation of G on Hilbert space $\mathbf{H}$ and a covariant mapping $\phi : X \to \mathbf{H}$ sending outcomes of X to unit vectors, in such a way that $x \perp y$ in X implies $\phi(x) \perp \phi(y)$ in $\mathbf{H}$. Indeed, one can simply take $\mathbf{H} = \ell^2(X)$ under the regular representation of G, and identify X with the latter's standard basis. This embedding is of limited interest, of course: one wants to reduce the dimension of $\mathbf{H}$ as much as possible. We shall show that, if $(X, \mathfrak{A})$ enjoys a fairly weak homogeneity property relative to a *compact* group, and has an invariant finite-dimensional, separating set Δ of states, then the dimension of $\mathbf{H}$ can be taken to be $n + 1$, where n is the affine dimension of Δ.

Suppose $(X, \mathfrak{A})$ is a G-test space. Let Δ denote a convex set of states on $(X, \mathfrak{A})$. Denote the linear span of Δ in $\mathbb{R}^X$ by $V(\Delta)$. This has dimension $n + 1$, n the affine dimension of Δ. For every outcome $x \in X$, we have a functional $f_x \in V(\Delta)^*$ given by $f_x(\omega) = \omega(x)$ for all $\omega \in \Delta$; we also have a unit functional $\mathbf{1} : V(\Delta) \to \mathbb{R}$ given by $\mathbf{1}(\omega) = 1$. Note that $\sum_{x\in E} f_x = \mathbf{1}$ (the sum converging in the weak-$*$ topology). The action of G on X gives rise to a linear action L of G on $V(\Delta)$, given by $L_a(\omega) = \omega(a^{-1}x)$ for all $x \in X$ and all $\omega \in V(\Delta)$, along with the dual representation L^* on $V(\Delta)^*$ given by $L^*_a(f) = f \circ L_a$ for all $a \in G$, $f \in V(\Delta)^*$. In the case of the frame manual $(X_{\mathbf{H}}, \mathfrak{F}_{\mathbf{H}})$ associated with a Hilbert space $\mathbf{H}$, if one takes for Δ the set of *all* states, then Gleason's Theorem lets us identify $V(\Delta)$ with the space of trace-class hermitian operators on $\mathbf{H}$, and $V^*(\Delta)$, with the space of all hermitian operators on $\mathbf{H}$. If $\mathbf{H}$ is finite dimensional, this last carries a natural inner product given by $\langle A, B\rangle = \mathrm{Tr}(AB^*)$, invariant under the natural action $A \mapsto U^*AU$ of $\mathbf{H}$'s unitary group.

Definition 5.1 Call a G-test space $(X, \mathfrak{A})$ 2-*transitive* iff G acts transitively on the orthogonality relation $\perp \subseteq X^2$—in other words, for any pairs of outcomes $x \perp y$ and $u \perp v$, there exists some $\alpha \in G$ with $u = \alpha x$ and $v = \alpha y$.

Evidently, any fully-symmetric G-test space is 2-transitive.

Theorem 5.2 *Let G be a compact group, let $(X, \mathfrak{A})$ be a 2-transitive uniform G-test space of rank $n < \infty$, and let Δ be a G-invariant, finite-dimensional, separating set of states on $(X, \mathfrak{A})$. Then there exists a G-invariant inner product on $V^*(\Delta)$ and a G-equivariant mapping $X \to V^*(\Delta)$ taking each $x \in X$ to a unit vector $p_x \in V^*(\Delta)$, such that $x \perp y \Rightarrow \langle p_x, p_y\rangle = 0$.*

Proof Let G act on $V^*(\Delta)$ as discussed above. Since G is compact, there exists a G-invariant inner product $\langle, \rangle$ on $V^*(\Delta)$ (see, e.g., [8]), which we normalize so that $\|\mathbf{1}\| = 1$. With respect to this inner product, the representation $a \mapsto L^*_a$ of G on $V^*(\Delta)$ is unitary. For each $x \in X$, set $q_x = f_x - \langle f_x, \mathbf{1}\rangle\mathbf{1}$, so that

$$\langle q_x, \mathbf{1}\rangle = 0. \tag{1}$$

Notice that $L^*_\alpha q_x = q_{\alpha x}$ for all $\alpha \in G$ and all $x \in X$. Since L is unitary and G acts transitively on X, the vectors q_x have a constant norm $\|q_x\| \equiv r$. Moreover, since G takes any orthogonal pair of outcomes to any other, $\theta := \langle q_x, q_y\rangle$ is constant for any pair $x \perp y$ in X. If $\theta = 0$, we are done: simply set $p_x = q_x/\|q_x\|$. If not, we have

$$0 = \langle q_x, 0\rangle = \left\langle q_x, \sum_{y\in E} q_y \right\rangle = r^2 + (n-1)\theta.$$

In particular, $\theta = -\frac{r^2}{n-1} < 0$. In this case, set $p_x = q_x + c\mathbf{1}$ where $c = \sqrt{-\theta}$. Then the mapping $x \mapsto p_x$ is a G-equivariant injection, and using (1), we have $\langle p_x, p_y \rangle = 0$ whenever $x \perp y$. Replacing p_x by $p_x/\|p_x\|$ if necessary finishes the proof. □

Note that the embedding of Theorem 5.2 will not, in general, preserve orthogonality in both directions. That is, it may happen that $p_x \perp p_y$ even though $x \not\perp y$ in X. Indeed, any fully symmetric test space having a 4-loop—e.g., any n-by-n grid—serves to illustrate this, since $\mathfrak{F}(\mathbf{H})$ itself has no 4-loops.

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